The Saga of the Four-Color Theorem

(+ other facts about graph coloring)

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ALGO seminar

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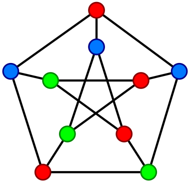
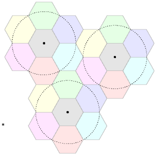
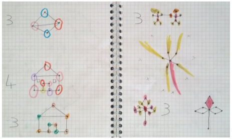
Graph coloring

Goal: Assign of a color to every vertex such that adjacent vertices have different colors.





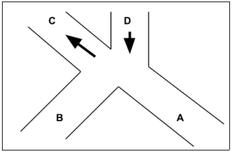


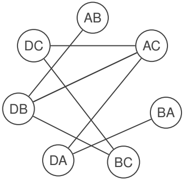
Many applications (e.g. telecom)

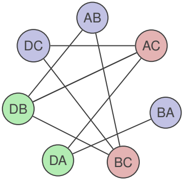
- Wireless communications
- In general, mutual exclusion, scheduling, ...
- Occupy a 5-y.o. kid

Graph coloring (2)

Example: traffic lights (credit: Eric Sopena)







1) Create a conflict graph G of the trajectories.

2) Color G

3) We obtain 3 *color classes*: $1 : \{AB, DC, BA\}$, $2 : \{DB, DA\}$, $3 : \{AC, BC\}$.

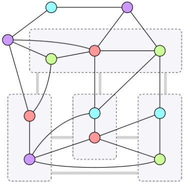
$\rightarrow$ All trajectories in the same class can have green light at the same time.

Further examples: time tables; altitude of aircrafts; anything to be optimized against *conflicts*.

Chromatic number $\chi(G)$

$\chi(G) =$ minimum number of colors needed in G.

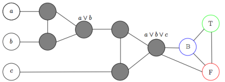
- At least the size of any *clique* in G
- Hadwiger's conjecture (1943):
  At most the size of a clique *minor* in G.



On the algorithmic side

Complexity of k-coloring

- 2-COL is linear (if and only if bipartite graph)
- 3-COL is NP-hard (reduction from SAT)
- 4-COL is NP-hard (reduction from 3-COL)
- k-COL is NP-hard (reduction from (k-1)-COL)

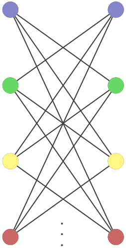


[Garey, Johnson, Stockmeyer, 1976]
(drawing: Yu Cheng)

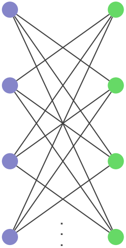
The `First-Fit` algorithm

For each vertex in G:
Try color 1, then 2, then 3...

Very fast, but arbitrarily far from optimum
(if we pick the vertices in bad order)



$O(n)$ colors



$O(1)$ colors

Still, essentially the best we can do!

(No $n^{1-\epsilon}$ approx in polynomial time... [Zuckermann, 2007])

The four color theorem

Every *planar* graph is four colorable *(planar = can be drawn without crossing edges).*

Timeline:

1852 Francis Guthrie (botanist) notices that **four** colors are enough to color the map of England's counties.

1879 Kempe proves the conjecture.

1880 Tait formulates it in terms of **planar graphs**, and gives a different proof.

1890 Heawood finds a bug in Kempe's proof and adapts it to prove that **five** colors are enough.

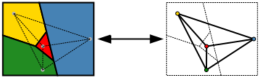
1891 Petersen finds a bug in Tait's proof.

1960s Heesch starts using computers to search for a proof

1976 Appel and Haken **succeed!**
$\rightarrow$ reduction from ∞ to 1834 possible configurations, all **checked by computer**.

1996 Robertson, Sanders, Seymour reduce it to 633 configurations.

2005 Gonthier certifies the proof using **Coq**. □



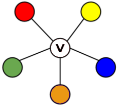
Proof of the 5 color theorem

Theorem: planar graphs are 5 colorable.

Warm-up: what about 6 colors first?

- Euler's formula: in planar graphs, $v - e + f = 2$ (v = # vertices, e = # edges, f = # faces)
- Implies (not immediate) that every planar graph has a vertex of degree ≤ 5
- Recursive algorithm:
  1. Find a vertex v of degree ≤ 5
  2. Color $G \setminus v$
  3. Give an available color to v (guaranteed by its degree)

  Base case: If G has $\leq$ 5 vertices, give a different color to each vertex.



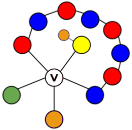
5 colors: Kempe's chains

Similar ideas as 6 colors, with an additional trick (by Kempe, 1879).

Step 2 becomes:
Color $G \setminus v$, then tweak the coloring so that the neighbors of v use at most 4 colors.

This is indeed always possible!

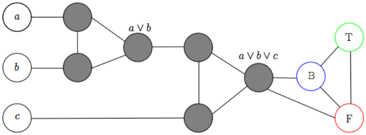
There must exist two colors which do not induce a connected component
$\rightarrow$ flip one of the components, one color is freed.



4 colors?

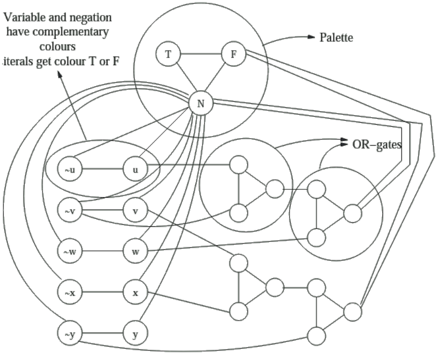
Not for today :-)

Reduction from 3-SAT to 3-COL



(credit: Lalla Mouatadid)

$$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$$



(credit: Igor Potapov)