Temporal Cliques admit Sparse Spanners (reloaded)

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[1] Joint work with:
- Joseph Peters and Jason Schoeters (ICALP 2019)
- **Daniele Carnevale and Timothée Corsini (SAND 2025)**.

+ Disjoint work by Angrick et al. (ESA 2024).

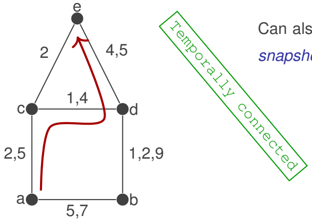
Temporal graphs

(a.k.a. time-varying, time-dependent, evolving, dynamic,...)

$\mathcal{G} = (\underline{V, E}, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *time labels* to edges.

footprint of $\mathcal{G}$

Example:



Can also be viewed as a sequence of *snapshots* $\{G_i = \{e \in E : i \in \lambda(e)\}\}$

Restrictions on labeling: *simple* ($\lambda : E \to \mathbb{N}$); *proper* ($\lambda$ locally injective), *happy* (both).

Temporal paths

- e.g. $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ (*strict*)
- e.g. $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ (*non-strict*)

Temporal connectivity: All-pairs reachability (class TC).

$\to$ Warning: In general, reachability is non-symmetrical... and non-transitive!

Spanning trees

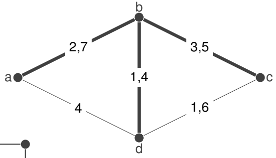
In static graphs

- Existence is guaranteed
- Size is always $n-1$

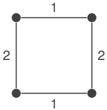
Temporal spanning tree ?

Input: A temporal graph $\mathcal{G} \in$ TC.

Goal: Find a spanning tree S of the *footprint*, so that $\mathcal{G}[S] \in$ TC.



Does not always exist:



In fact, NP-hard to decide! [Casteigts, Corsini, 2024]

Searching for the lost tree

What to replace trees?

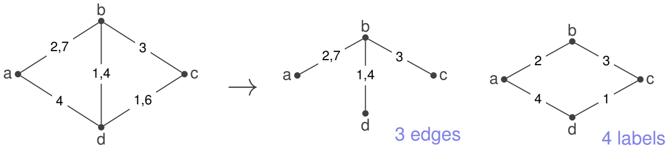
→ Small reachability substructures (*temporal spanners*).

Temporal spanners

Input: a temporal graph $\mathcal{G} \in$ TC

Output: a temporal subgraph $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{G}' \in$ TC

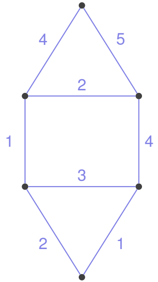
Cost measure: # edges or # labels



Complexity:

- MIN-EDGE (and MIN-LABEL): APX-hard for simple, non-proper, non-strict [Axiotis, Fotakis, 2016]
- MIN-LABEL: APX-hard for non-simple, non-proper, strict [Akrida, Gasieniec, Mertzios, Spirakis, 2017]
- MIN-EDGE: NP-hard for non-simple, proper [Casteigts, Corsini, 2024]

From this point on, all temporal graphs are **happy**



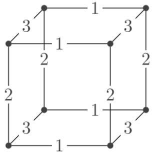
- Simple
- Proper

Approved by Pharrell W.:

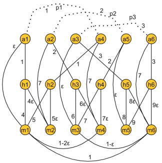
Structural results

Given a temporal graph $\mathcal{G}$ that is temporally connected ($\mathcal{G} \in$ TC), is there any guarantee on the size of a minimum spanner $\mathcal{G}' \subseteq \mathcal{G}$?

Note: The absolute minimum is $2n - 4$ [Bumby, 1979 (gossip theory)]



- Are spanners of size $O(n)$ always guaranteed?
  $\rightarrow$ Nope, some hypercubes are minimal with $\Theta(n \log n)$ edges [Kleinberg, Kempe, Kumar, 2000]



- Are spanners of size $o(n^2)$ always guaranteed?
  $\rightarrow$ Not even! [Axiotis, Fotakis, 2016]

Any positive results?

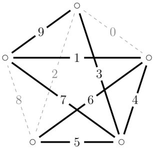
Good news 1 (probabilistic): [Casteigts, Raskin, Renken, Zamaraev, 2021]:

- Nearly optimal spanners (of size $2n + o(n)$) almost surely exist in **random** temporal graphs, and so, **as soon as** the graph becomes TC!

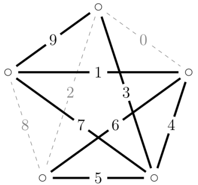
Good news 2 (deterministic): [Casteigts, Peters, Schoeters, 2019]:

- Spanners of size $O(n \log n)$ always exist in temporal **cliques**.
  Achieved using **dismountability** + a number of other techniques.

  This talk: dismountability is all you need! [Carnevale, Casteigts, Corsini, 2025]



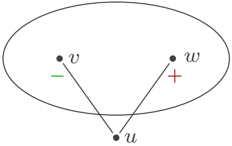
Temporal cliques admit $O(n \log n)$ spanners



(1-hop) dismountability

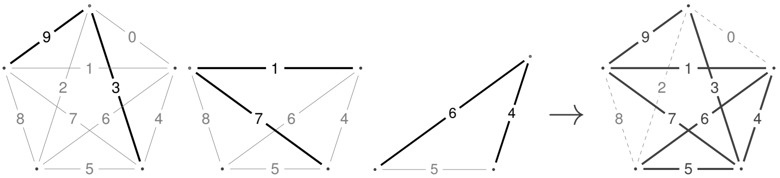
Find a node u s.t. :

- $uv =$ minimum edge of some v (denoted $e^-(v)$)
- $uw =$ maximum edge of some w (denoted $e^+(w)$)



Then $\texttt{spanner}(\mathcal{G})$:= $\texttt{spanner}(\mathcal{G}[V \setminus u])$ + uv + uw

$\rightarrow$ Recurse.

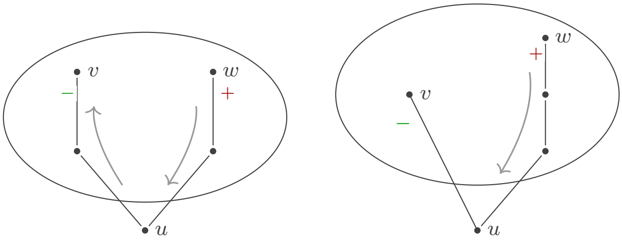


If applicable recursively, gives spanner of size $2n - 3$.

Unfortunately, not always applicable :-(

Relaxed version: k-hop dismountability

Temporal paths $u \rightsquigarrow v$ ending at $e^-(v)$ and $w \rightsquigarrow u$ starting at $e^+(w)$



(a) Example of 2-hop dismountable

(b) Example of 3-hop dismountable

$\rightarrow$ select both paths in the spanner

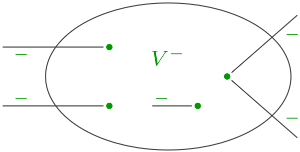
$\rightarrow$ recurse! (in $\mathcal{G} \setminus u$)

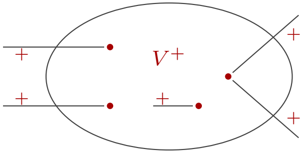
If applicable recursively for some $k = O(1)$, we get a $O(n)$ spanner.

Again, not always feasible, but...

The absence of dismountability gives rise to an interesting structure.

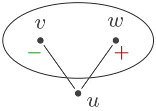
Non 1-hop dismountable cliques





$V^- = \{u \in V : uv = e^-(v) \text{ for some } v\}$ $\quad$ $V^+ = \{u \in V : uv = e^+(v) \text{ for some } v\}$

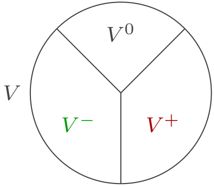
$V^0 =$ rest of the nodes.



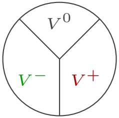
What if V^- and V^+ overlap?

$\implies \exists u \in V^- \cap V^+$, so u is 1-hop dismountable! $\rightarrow$ recurse.

If the clique is **non 1-hop dismountable**, then
V^-, V^+, and V^0 must **partition** V.

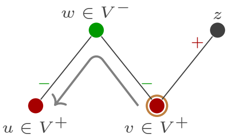


Non $\{1,2\}$-hop dismountable cliques



Thm: If the minimum edge of two or more vertices in V^+ go to a **same** vertex in V^-, then the graph is 2-hop dismountable.

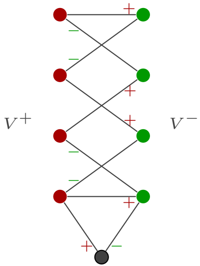
The same holds for maximum edges of vertices in V^-.



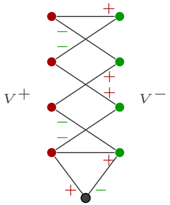
Consequence: non $\{1,2\}$-hop dismountable cliques satisfy:

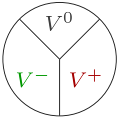
- The edges $\{e^-(v) : v \in V^+\}$ form a *matching*.
- The edges $\{e^+(v) : v \in V^-\}$ form a *matching*.
- V^- and V^+ have *equal size*.

Example:

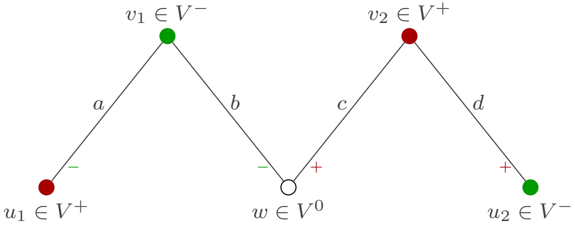


Non $\{1,2\}$-hop dismountable cliques





What about V^0 ?



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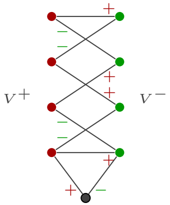

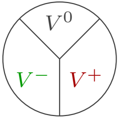

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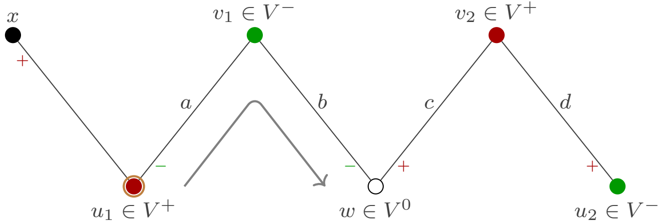
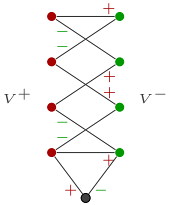
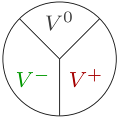


Non $\{1,2\}$-hop dismountable cliques





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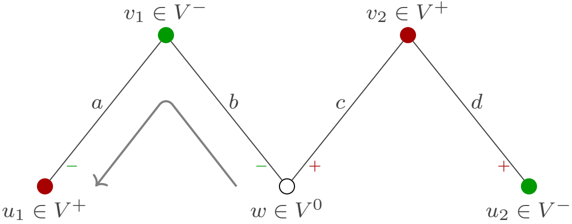
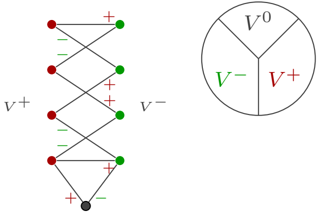


Non $\{1,2\}$-hop dismountable cliques



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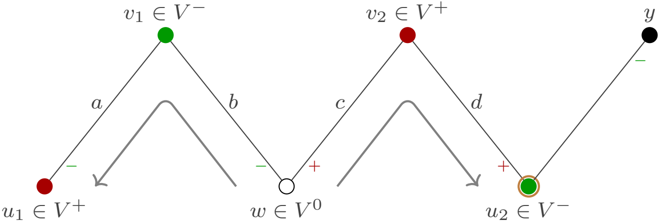
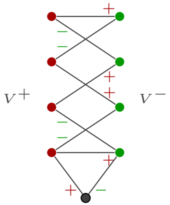
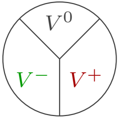


Non $\{1,2\}$-hop dismountable cliques





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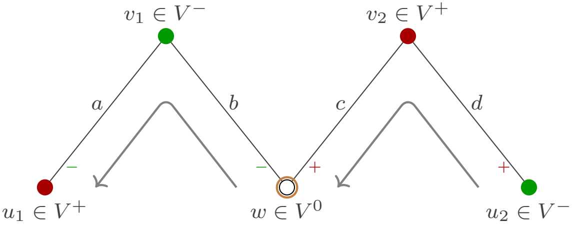
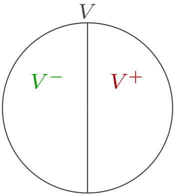
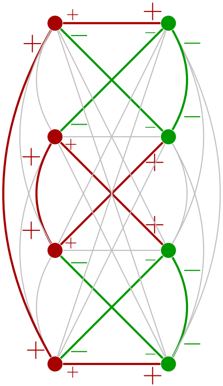


If $\mathcal{G}$ is non $\{1,2\}$-hop dismountable, then V^0 **is empty**!

Summary of non $\{1,2\}$-hop dismountability



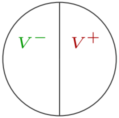


If $\mathcal{G}$ is non $\{1,2\}$-hop dismountable, then:

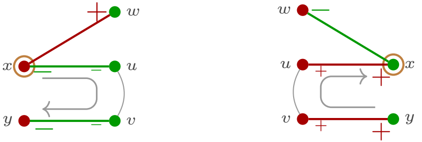
1. V^- and V^+ are the same size and **partition** of V.
2. The set $M^- := \{e^-(v) : v \in V^+\}$ is a **perfect matching**.
3. The set $M^+ := \{e^+(v) : v \in V^-\}$ is a **perfect matching**.

(If fact, if and only if)

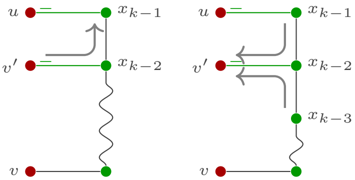
Non $\{1,2,3\}$-hop dismountable cliques



A non $\{1,2\}$-hop dismountable clique is 3-hop dismountable if and only if we have such temporal paths:



Theorem 3,7
k-hop dismountable $\implies$ $\{1,2,3\}$-hop dismountable



$\implies$ We can stop the analysis at $k = 3$.

$\implies$ Any **minimal counter-example** to the existence of $4n$ spanners must have all the properties of non $\{1,2,3\}$-hop dismountability.

Exploiting the structure

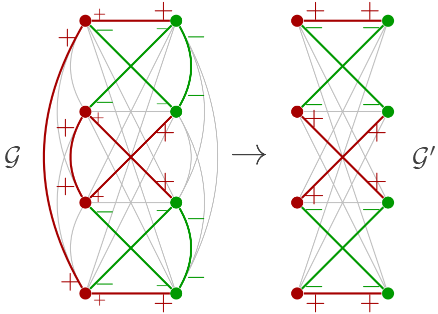
As far as $O(n)$ spanners are concerned (let apart the constant), excluding $\{1, 2\}$-hop dismountability is sufficient.

Why?

- Let $\mathcal{G}' \subseteq \mathcal{G}$ be the bipartite part between V^- and V^+.
- $\mathcal{G}'$ is **extremally matched** (reciprocal $-$ and $+$ edges)
- $\mathcal{G}' \in$ TC
- Any spanner of $\mathcal{G}'$ is a spanner of $\mathcal{G}$

Thm: Extremally matched bicliques admit $O(n)$ spanners **if and only if** temporal cliques admit $O(n)$ spanners.

($\Longrightarrow$) [Casteigts, Peters, Schoeters, 2019]
($\Longleftarrow$) [Angrick et al., 2024]



Let's work in extremally matched temporal bicliques!

Remarks:

1. We can add the two matchings to the spanner (essentially free)
2. Focus on preserving reachability from **left** to **right** only
   (together with the matchings, this guarantees the spanner is TC)

Can we find a $O(n)$ spanner in this setting? (Spoiler alert: We still don't know (open problem)... but)

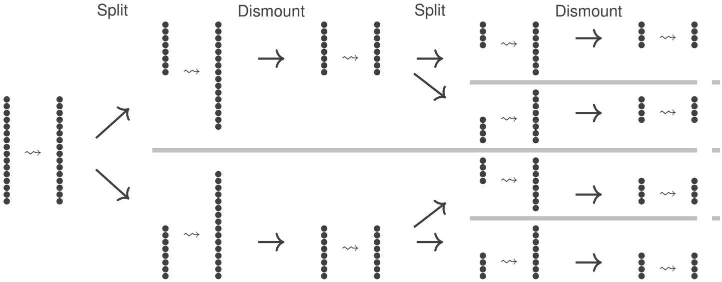
$O(n \log n)$ spanners using one-sided dismountability

Input: Extremally matched temporal biclique.

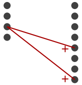
Goal: Find a spanner that preserves Left-to-Right reachability.

Thm: $O(n \log n)$ spanners always exist [Casteigts, Peters, Shoeters, 2019]

$\downarrow$ A much simpler proof by [Angrick et al., 2024]



1. Split the work, achieving both halves of V^+ to all of V^- separately.
2. Dismount vertices of V^- whose $+$ collide in V^+ (pay two edges).
3. Recurse.



$$cost(n) = 2 \cdot cost(n/2) + O(n)$$

By the Master's theorem for recurrences, the total cost is $\boxed{O(n \log n)}$. □

Open questions

Algorithmic

- Complexity of MIN-SPANNER in happy graphs?

Structural

- Do temporal cliques admit spanners of size $O(n)$?
- Do temporal cliques admit spanners of size $2n - 3$? (at least true for $n \leq 8$)
- Beyond temporal cliques?

Thanks!