Temporal graph theory: paradigm and algorithmic challenges

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Séminaire “Opérations”,
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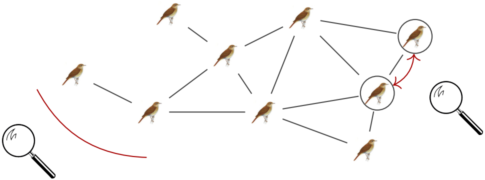
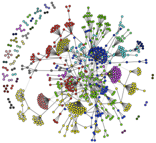
Theory of networks



Network as data

→ centralized algorithms...



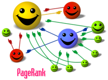
Network as environment

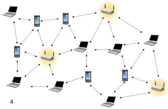
→ decentralized algorithms...
(a.k.a. distributed)

The world is dynamic...

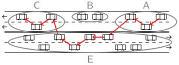
In technologies

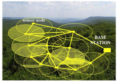


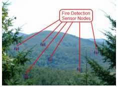


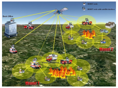


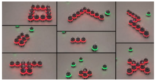
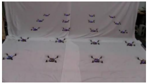










In nature

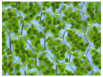
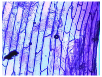






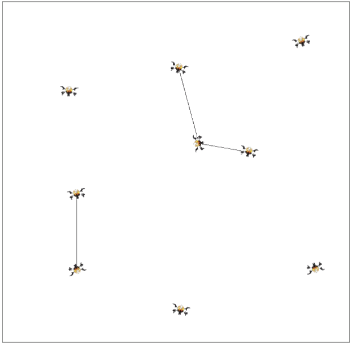




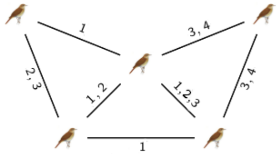



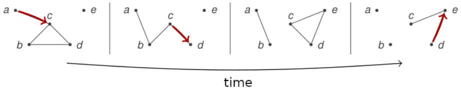

(Highly) dynamic networks?

Example of scenario



Modeling



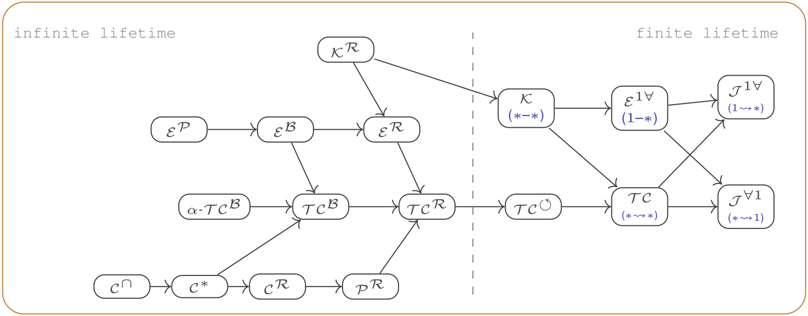


Properties:

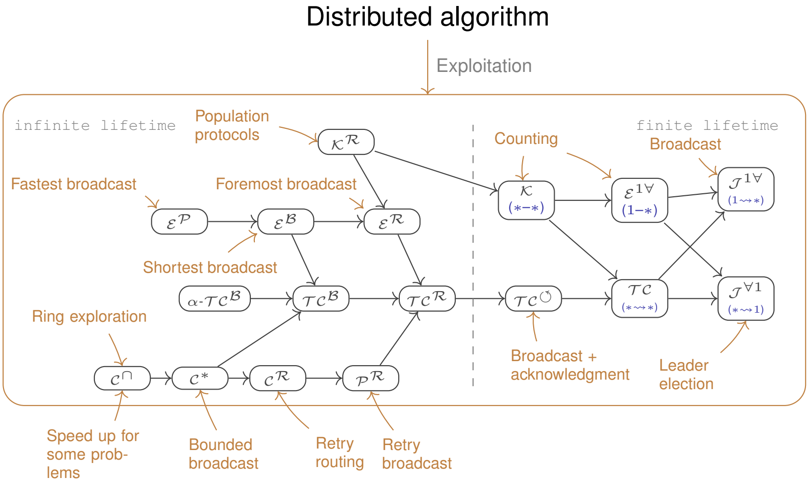
- Temporal connectivity? $\mathcal{TC}$
- Repeatedly? $\mathcal{TC}^{\mathcal{R}}$
- Recurrent links? $\mathcal{E}^{\mathcal{R}}$
- In bounded time? $\mathcal{E}^{\mathcal{B}}$
- ...

$\rightarrow$ Classes of temporal graphs

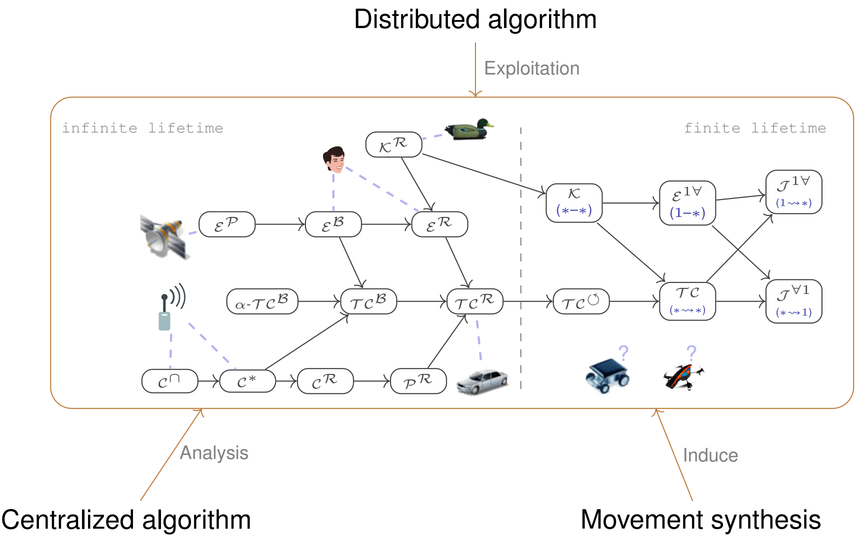
Some classes of temporal graphs



Some classes of temporal graphs



Some classes of temporal graphs

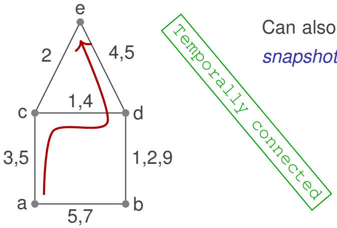


Temporal graphs
(*a.k.a.* time-varying, time-dependent, evolving, dynamic,...)

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *time labels* to edges.

footprint of $\mathcal{G}$

Example:



Can also be viewed as a sequence of *snapshots* $\{G_i = \{e \in E : i \in \lambda(e)\}\}$

Temporal paths

- e.g. $\langle (a, c, 3), (c, d, 4), (d, e, 5) \rangle$ (*strict*)
- e.g. $\langle (a, c, 3), (c, d, 4), (d, e, 4) \rangle$ (*non-strict*)

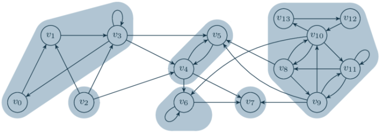
Temporal connectivity: Temporal paths between all vertices.

$\to$ Warning: Reachability is non-symmetrical... and non-transitive!

Some restrictions: *simple* ($\lambda : E \to \mathbb{N}$); *proper* ($\lambda$ locally injective), *happy* (both).

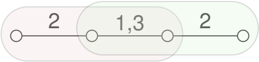
Impact of non-transitivity (Example: connected components)

In static graphs



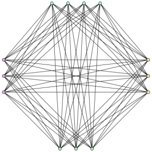
- Components define a partition
- Easy to compute

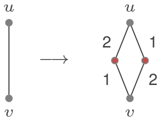
In temporal graphs



- Maximal components may overlap
- Can be exponentially many

MAX COMPONENT is NP-hard! (from CLIQUE) Bhadra, Ferreira, 2003





- Replace edges with semaphore gadgets
- Cliques become temporal components

Spanning trees

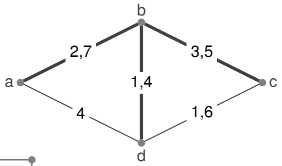
In static graphs

Spanning tree:

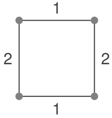
- Existence is guaranteed
- Size is always $n - 1$

In temporal graphs ?

$\rightarrow$ Restrict the footprint to a spanning tree, while *preserving* temporal connectivity.



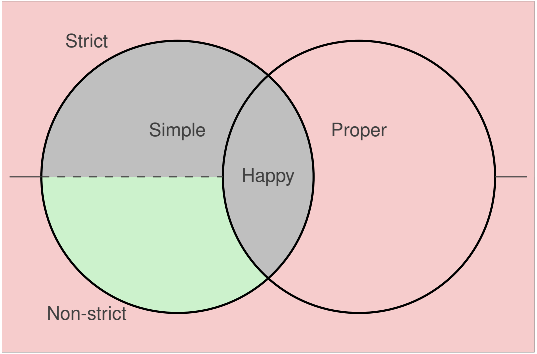
Does not always exist:



In fact, NP-hard to decide!

Hardness of Spanning trees

[C., Corsini (SIROCCO 2024)]



NP-Hard / Polynomial / Impossible

Searching for the lost tree

How to relax the definition?

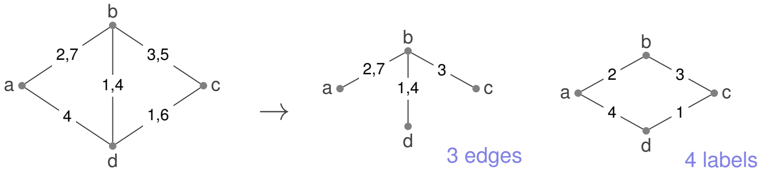
$\rightarrow$ "small" temporal spanners

Temporal spanners

Input: a temporally connected graph $\mathcal{G}$ ($\mathcal{G} \in TC$)

Output: a temporal subgraph $\mathcal{G}' \subseteq \mathcal{G}$ that preserves reachability ($\mathcal{G}' \in TC$)

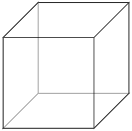
Cost measure: # edges or # labels

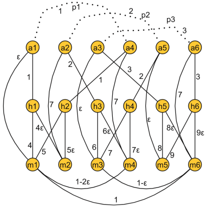


Bad news

Structural results

- Size $O(n)$? Nope
  (Kleinberg, Kempe, Kumar, 2000)
- Size $o(n^2)$? Nope
  (Axiotis, Fotakis, 2016)





Complexity

- MIN-LABEL: APX-hard (non-simple, non-proper, strict) (Akrida, Gasieniec, Mertzios, Spirakis, 2017)
- MIN-EDGE (and MIN-LABEL): APX-hard (simple, non-proper, non-strict) (Axiotis, Fotakis, 2016)
- Open in *happy* graphs (i.e. simple and proper).

Beyond size (positive and negative)

- Distance-preserving (Bilò, D'Angelo, Gualà, Leucci, Rossi, 2022)
- Fault-tolerant (Bilò, D'Angelo, Gualà, Leucci, Rossi, 2022)

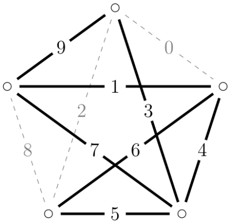
Good news

Good news 1: (C., Raskin, Renken, Zamaraev, FOCS 2021):

- Nearly optimal spanners (of size $2n + o(n)$) almost surely exist in **random** temporal graphs, as soon as the graph becomes temporally connected

Good news 2: (C., Peters, Schoeters, ICALP 2019):

- Spanners of size $O(n \log n)$ always exist in temporal **cliques**

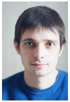


Good news 1:

Spanners of size $2n + o(n)$ almost surely exist

in random temporal graphs

(with)

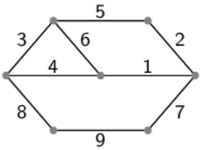


Connectivity in random temporal graphs

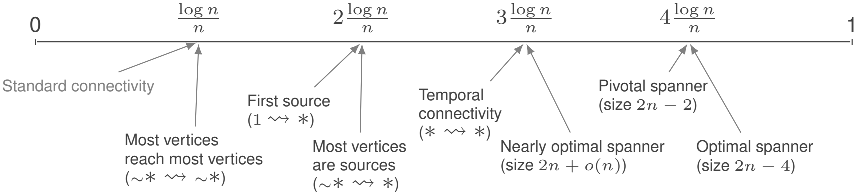
(C., Raskin, Renken, Zamaraev, 2021)

Random happy temporal graphs:

1. Pick an Erdös-Rényi $G \sim G_{n,p}$
2. Permute the edges randomly, interpret as (unique) presence time



Timeline for p (as $n \to \infty$):

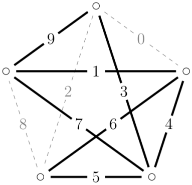


All the thresholds are sharp.

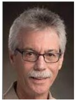
(sharp: $\exists \epsilon(n) = o(1)$, not true at $(1 - \epsilon(n))p$, true at $(1 + \epsilon(n))p$)

Good news 2:

Temporal cliques admit sparse spanners



(with)



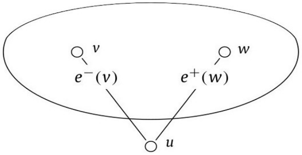
Dismountability

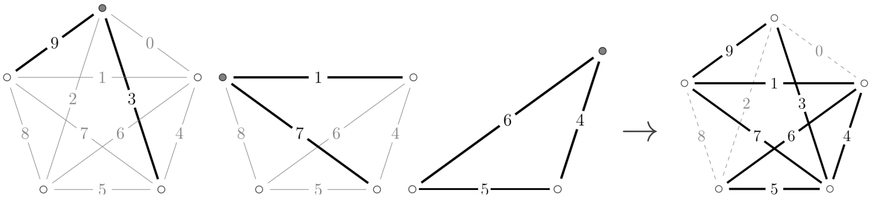
Find a node u s.t. :

- $uv =$ min edge of v
- $uw =$ max edge of w

Then $\texttt{spanner}(\mathcal{G})$:= $\texttt{spanner}(\mathcal{G}[V \setminus u])$ + uv + uw

$\rightarrow$ Recurse.





Not always feasible.

spanner of size $2n - 3$.

Relaxed version: k-hop dismountability

- Temporal paths $u \rightsquigarrow v$ ending at $e^{-}(v)$ and $w \rightsquigarrow u$ starting at $e^{+}(w)$

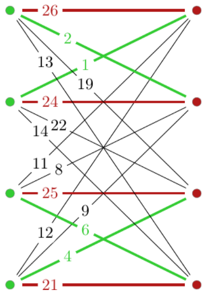
Select these $2k$ edges, then recurse $\rightarrow$ $O(n)$-spanner if k constant.

Not always feasible, but...

What if dismountability fails?

If $\mathcal{G}$ is neither 1-hop nor 2-hop dismountable, then the following is guaranteed:

- Complete bipartite graph $\mathcal{H} \subseteq \mathcal{G}$ ($n/2$ vertices in each part)
- Min edges of green nodes form a matching
- Max edges of red nodes form a matching
- Both matchings are disjoint
- **A spanner of $\mathcal{H}$ is a spanner of $\mathcal{G}$**



New goal: $\rightarrow$ Sparsify $\mathcal{H}$.

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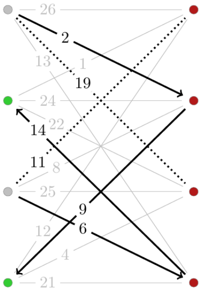


New goal: $\to$ Sparsify $\mathcal{H}$.

Main lemma:

Half of the green vertices can be iteratively removed, at doubling cost.
Repeat $\log n$ times.

$\to$ Spanners of size $O(n \log n)$ always exist.

Open questions on spanners

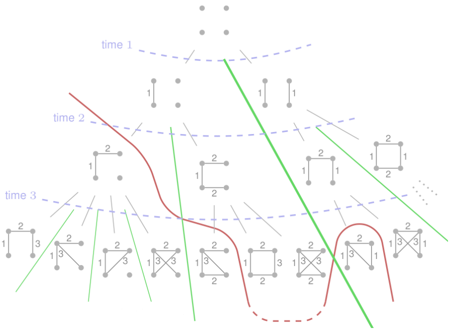
Algorithmic

- Complexity of MIN-SPANNER in happy graphs?

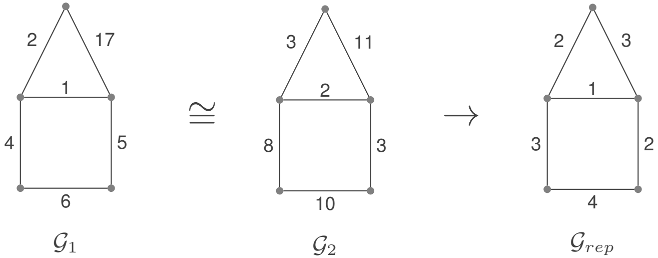
Structural

- Do cliques admit spanners of size $O(n)$?
- Do cliques admit spanners of size $2n - 3$?
- What else than cliques?

Enumeration of happy temporal graphs



Equivalence based on reachability



How to capture this equivalence?

- Canonical representatives

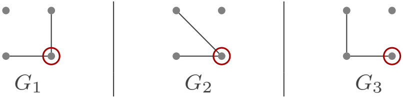
Good news:

- Finite number
- Canonization, isomorphism testing, and automorphism generators all computable in *polynomial time*.

Miscellaneous (1)

Combinatorial problems revisited

E.g.: Covering problems like DOMINATING SET



$\rightarrow$ *Temporal* variant

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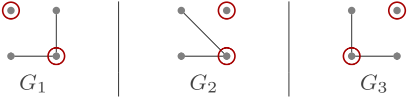

→ *Temporal* variant

→ *Evolving* variant (a.k.a. “dynamic graph algorithms”)

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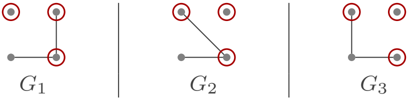

→ *Temporal* variant

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→ *Permanent* variant

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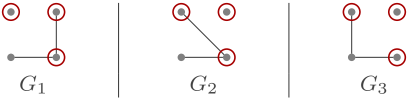


→ *Temporal* variant

→ *Evolving* variant (a.k.a. “dynamic graph algorithms”)

→ *Permanent* variant

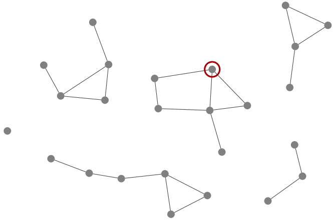
$PermanentDS \supsetneq EvolvingDS_i \supsetneq TemporalDS.$

+ sliding windows versions (many results recently).

Miscellaneous (2)

Distributed problems revisited

E.g.: Election, Spanning trees, ...



How are they defined?

$\rightarrow$ Several options

- One global leader, elected once and forever

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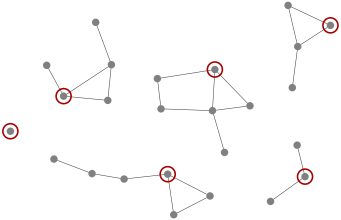

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- One global leader, elected once and forever
- One leader per component, updated as the graph changes

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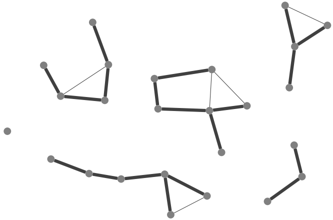

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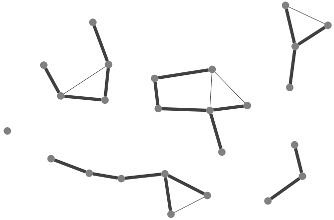

How are they defined?

→ Several options

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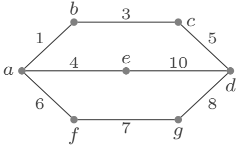
- One leader per component, updated as the graph changes

Both are very different in essence!

Miscellaneous (3)

Optimal temporal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]

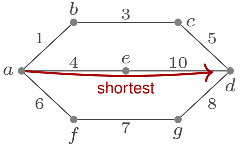


Which way is optimal from a to d?

Miscellaneous (3)

Optimal temporal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]



Which way is optimal from a to d?

-min hop?

Miscellaneous (3)

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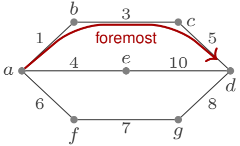


Which way is optimal from a to d?

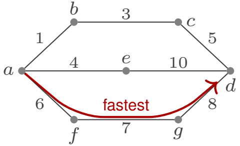
-min hop?

-earliest arrival?

Miscellaneous (3)

Optimal temporal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]



Which way is optimal from a to d?

-min hop?

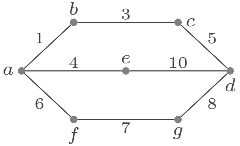
-earliest arrival?

-fastest traversal?

Miscellaneous (3)

Optimal temporal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]



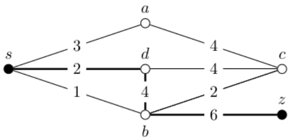
Which way is optimal from a to d?

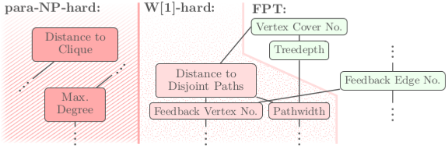
-min hop?

-earliest arrival?

-fastest traversal?

Waiting constraints





[C., Himmel, Molter, Zschoche, 2021]

Goal: define new parameters which are themselves temporal!

Thanks!

G. Stauffer (Sep 19, 2023) :-)