Temporal graph theory:
paradigm and algorithmic challenges

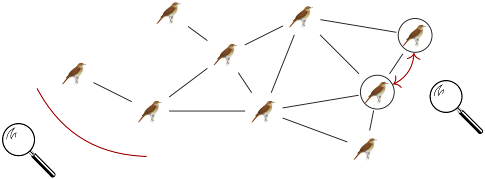
Arnaud Casteigts

University of Geneva

June 26, 2024

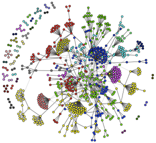
Swiss OR days,
Haute École de Gestion (Geneva)

Studying networks



Network as data

→ centralized algorithms...

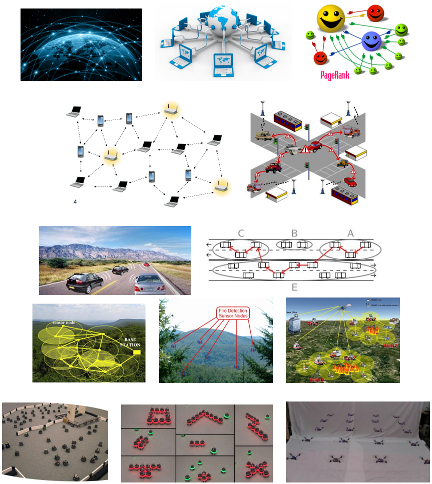


Network as environment

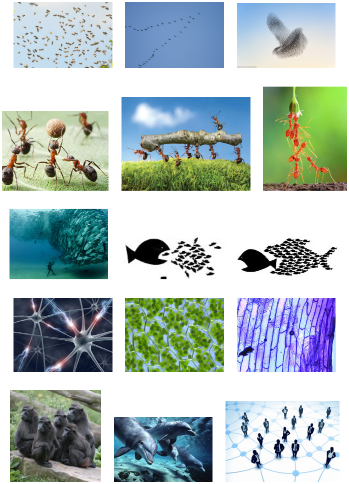
→ decentralized algorithms...
(a.k.a. distributed)

The world is dynamic...

In technologies

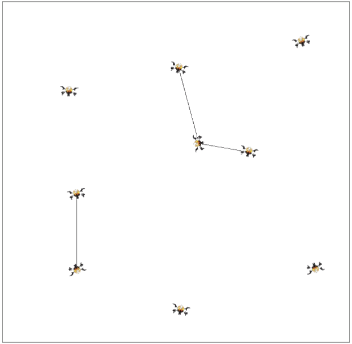


In nature

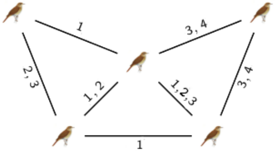


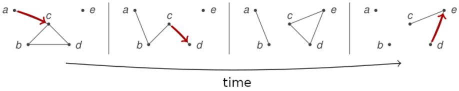
(Highly) dynamic networks?

Example of scenario



Modeling





Properties:

- Temporal connectivity? $\mathcal{TC}$
- Repeatedly? $\mathcal{TC}^{\mathcal{R}}$
- Recurrent links? $\mathcal{E}^{\mathcal{R}}$
- In bounded time? $\mathcal{E}^{\mathcal{B}}$
- ...

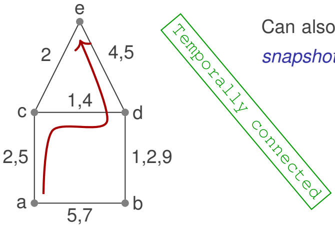
→ Classes of temporal graphs

Temporal graphs

(*a.k.a.* time-varying, time-dependent, evolving, dynamic,...)

$\mathcal{G} = (\underbrace{V, E}_{\textit{footprint} \text{ of } \mathcal{G}}, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *time labels* to edges.

Example:



Can also be viewed as a sequence of *snapshots* $\{G_i = \{e \in E : i \in \lambda(e)\}\}$

Special types of graphs: *simple* ($\lambda : E \to \mathbb{N}$); *proper* ($\lambda$ locally injective), *happy* (both).

Temporal paths

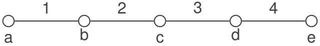
- e.g. $\langle(a, c, 2), (c, d, 4), (d, e, 5)\rangle$ (*strict*)
- e.g. $\langle(a, c, 2), (c, d, 4), (d, e, 4)\rangle$ (*non-strict*)

Temporal connectivity: All-pairs reachability (class TC).

→ Warning: In general, reachability is non-symmetrical... and non-transitive!

Time versus structure (basic observations)

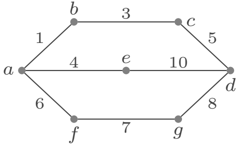
Centrality?



(time $\bmod 7$)

Optimal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]



Which path is optimal from a to d?

-min hop?

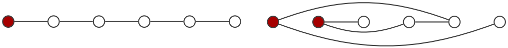
-earliest arrival?

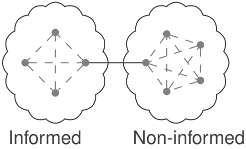
-fastest traversal?

Diameter of the snapshots versus propagation time

– Small diameter / Large propagation time ✓ $(\rightarrow)$

– Large diameter / Small propagation time ✓ $(\downarrow)$

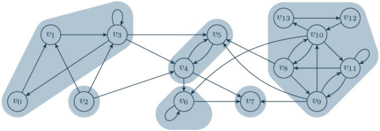




Impact of non-transitivity

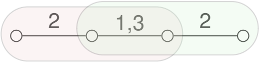
(Example: CONNECTED COMPONENTS)

In static graphs



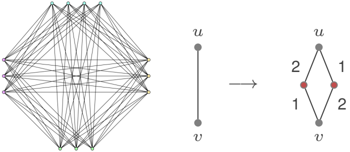
- Components define a partition
- Easy to compute

In temporal graphs



- Maximal components may overlap
- Can be exponentially many

COMPONENT is NP-hard! (from CLIQUE) [Bhadra, Ferreira, 2003]



- Replace edges with semaphore gadgets
- Cliques become temporal components

Spanning trees

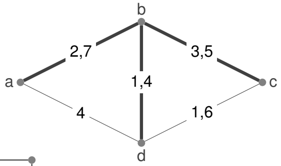
In static graphs

- Existence is guaranteed
- Size is always $n-1$

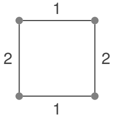
Temporal spanning tree ?

Input: A temporal graph $\mathcal{G} \in$ TC.

Goal: Find a spanning tree S of the *footprint*, so that $\mathcal{G}[S] \in$ TC.



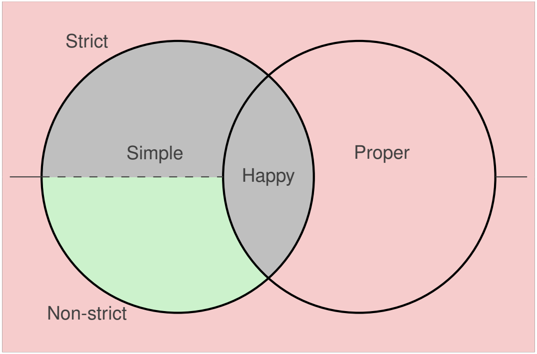
Does not always exist:



In fact, NP-hard to decide!

[C., Corsini, 2024]

Lanscape of hardness for SPANNING TREE



NP-Hard / Polynomial / Always no

Searching for the lost tree

What to replace trees?

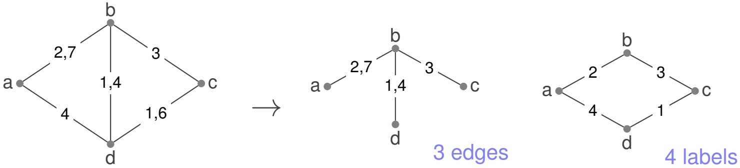
$\rightarrow$ Small reachability substructures (*temporal spanners*).

Temporal spanners

Input: a temporal graph $\mathcal{G} \in \mathsf{TC}$

Output: a temporal subgraph $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{G}' \in \mathsf{TC}$

Cost measure: # edges or # labels

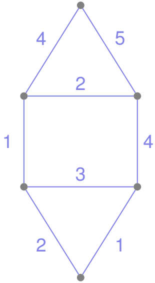


Complexity:

- MIN-LABEL: APX-hard for non-simple, non-proper, strict [Akrida, Gasieniec, Mertzios, Spirakis, 2017]
- MIN-EDGE (and MIN-LABEL): APX-hard for simple, non-proper, non-strict [Axiotis, Fotakis, 2016]

Open for happy graphs (i.e. both simple and proper), could be polynomial.

From this point on, all temporal graphs are **happy**



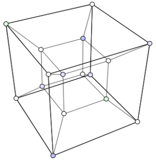
- Simple $\implies$ MIN-LABEL = MIN-EDGE
- Proper $\implies$ strict = non-strict
- Good prototype
- (Almost) no loss of generality

Approved by
Pharrell W.:

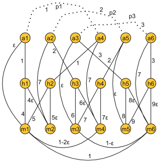
Structural results

Given a temporal graph $\mathcal{G}$ that is temporally connected ($\mathcal{G} \in$ TC), is there any guarantee on the size of a minimum spanner $\mathcal{G}' \subseteq \mathcal{G}$?

Note: The absolute minimum is $2n - 4$ [Bumby, 1979 (gossip theory)]



- Are spanners of size $O(n)$ always guaranteed?
  → Nope, hypercubes may fail [Kleinberg, Kempe, Kumar, 2000]
- Are spanners of size $o(n^2)$ always guaranteed?
  → Not even! [Axiotis, Fotakis, 2016]



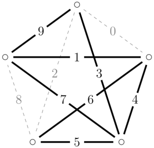
Any positive results?

Good news 1 (probabilistic): [C., Raskin, Renken, Zamaraev, 2021]:

- Nearly optimal spanners (of size $2n + o(n)$) almost surely exist in **random** temporal graphs, and so, **as soon as** the graph becomes TC!

Good news 2 (deterministic): [C., Peters, Schoeters, 2019]:

- Spanners of size $O(n \log n)$ always exist in temporal **cliques**



Open: $O(n)$ in temporal cliques?

Good news 1:

Spanners of size $2n + o(n)$ almost surely exist

in random temporal graphs

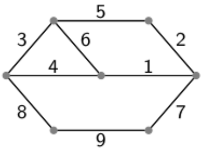
(with)

Connectivity in random temporal graphs

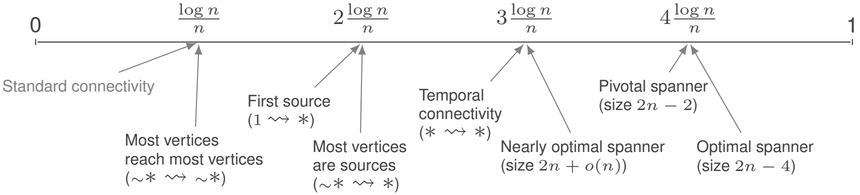
(C., Raskin, Renken, Zamaraev, 2021)

Random happy temporal graphs:

1. Pick an Erdös-Rényi $G \sim G_{n,p}$
2. Permute the edges randomly, interpret as (unique) presence time



Timeline for p (as $n \to \infty$):



0

$\frac{\log n}{n}$

$2\frac{\log n}{n}$

$3\frac{\log n}{n}$

$4\frac{\log n}{n}$

1

Standard connectivity

Most vertices reach most vertices ($\sim\!* \rightsquigarrow \sim\!*$)

First source ($1 \rightsquigarrow *$)

Most vertices are sources ($\sim\!* \rightsquigarrow *$)

Temporal connectivity ($* \rightsquigarrow *$)

Nearly optimal spanner (size $2n + o(n)$)

Pivotal spanner (size $2n - 2$)

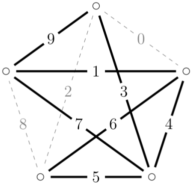
Optimal spanner (size $2n - 4$)

All the thresholds are sharp.

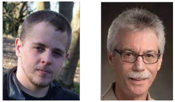
(sharp: $\exists \epsilon(n) = o(1)$, not true at $(1-\epsilon(n))p$, true at $(1+\epsilon(n))p$)

Good news 2:

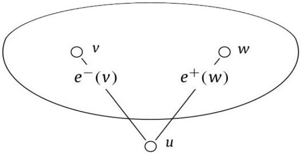
Temporal cliques admit sparse spanners



(with)



Dismountability

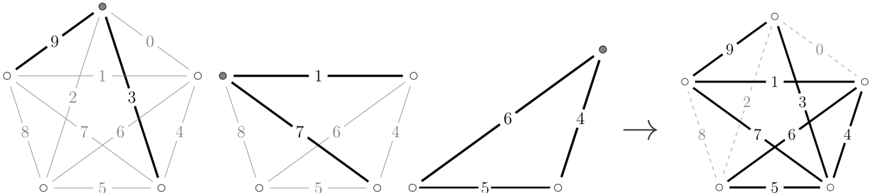


Find a node u s.t. :

- $uv =$ min edge of v
- $uw =$ max edge of w

Then $\texttt{spanner}(\mathcal{G})$:= $\texttt{spanner}(\mathcal{G}[V \setminus u])$ + uv + uw

$\rightarrow$ Recurse.



Not always feasible.

spanner of size $2n - 3$.

Relaxed version: k-hop dismountability

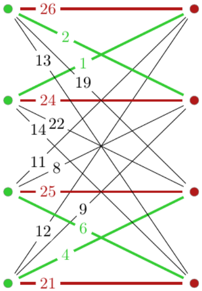
- Temporal paths $u \rightsquigarrow v$ ending at $e^-(v)$ and $w \rightsquigarrow u$ starting at $e^+(w)$

Select these $2k$ edges, then recurse $\rightarrow$ $O(n)$-spanner if k constant.

Not always feasible, but...

What if dismountability fails?

If $\mathcal{G}$ is neither 1-hop nor 2-hop dismountable, then the following is guaranteed:

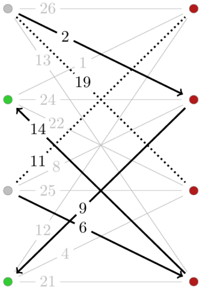


- Complete bipartite graph $\mathcal{H} \subseteq \mathcal{G}$ ($n/2$ vertices in each part)
- Min edges of green nodes form a matching
- Max edges of red nodes form a matching
- Both matchings are disjoint
- **A spanner of $\mathcal{H}$ is a spanner of $\mathcal{G}$**

New goal: → Sparsify $\mathcal{H}$.

What if dismountability fails?

If $\mathcal{G}$ is neither 1-hop nor 2-hop dismountable, then the following is guaranteed:



- Complete bipartite graph $\mathcal{H} \subseteq \mathcal{G}$ ($n/2$ vertices in each part)
- Min edges of green nodes form a matching
- Max edges of red nodes form a matching
- Both matchings are disjoint
- **A spanner of $\mathcal{H}$ is a spanner of $\mathcal{G}$**

New goal: → Sparsify $\mathcal{H}$.

Main lemma:

Half of the green vertices can be iteratively removed, at doubling cost.
Repeat $\log n$ times.

→ Spanners of size $O(n \log n)$ always exist.

Open questions on spanners

Algorithmic

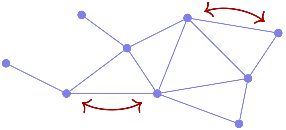
- Complexity of MIN-SPANNER in happy graphs?

Structural

- Do cliques admit spanners of size $O(n)$?
- Do cliques admit spanners of size $2n - 3$?
- What else than cliques?

Distributed Algorithms

(Think globally, act locally)



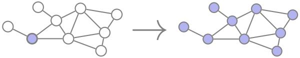
Collaboration of distinct entities to perform a common task.

No centralization available, interactions among neighbors.

Theoretical aspects of collective intelligence.

Examples of problems:

Broadcast



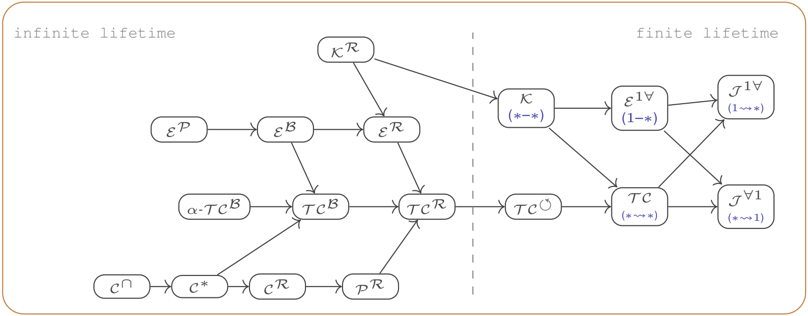
Spanning tree

Election

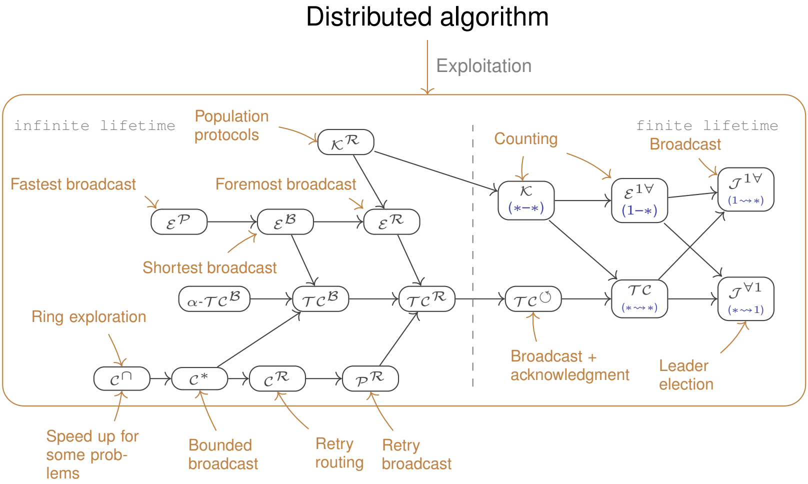
Counting

Consensus, naming, routing, exploration, coloring, dominating sets, ...

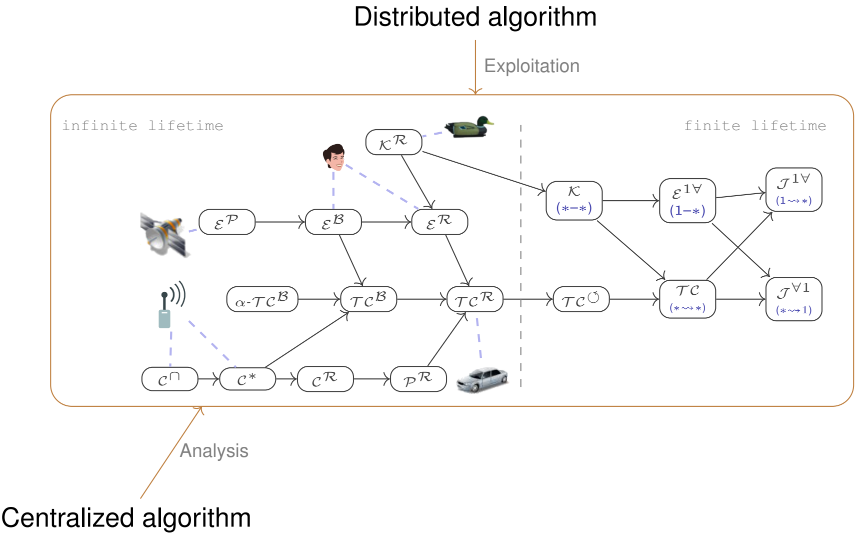
Some classes of temporal graphs



Some classes of temporal graphs



Some classes of temporal graphs



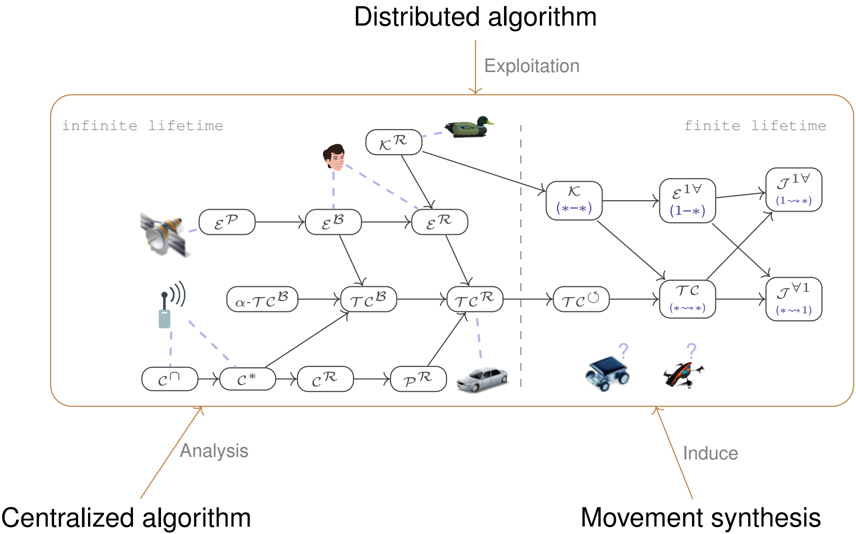
Distributed algorithm

Exploitation

Analysis

Centralized algorithm

Some classes of temporal graphs



Distributed algorithm

Exploitation

Analysis

Centralized algorithm

Induce

Movement synthesis

Thanks!

(Battelle, Dec 3, 2023)