Deterministic Leader Election in $O(D + \log n)$ Rounds with Messages of size $O(1)$

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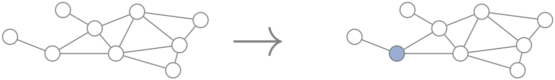
Leader election

“[Likely one of] the two most studied tasks in distributed computing literature” (Dinitz et al. JACM). More than 1500 papers published on this problem.

Def.: A distributed algorithm solves the election problem if it always terminates and in the final configuration exactly one process is marked as *elected* and all others are *non-elected*.



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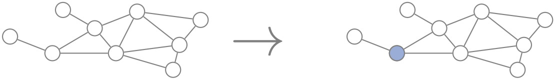

3 directions

- Feasibility of deterministic LE in anonymous graphs
- Complexity of deterministic LE with unique identifiers
- Complexity of randomized LE (in anonymous graphs)

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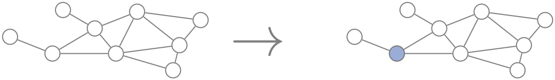

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This talk.

Model and complexity measures

The model

- Synchronous rounds (and simultaneous wakeup)
  1. Send messages to neighbors
  2. Receive messages from neighbors
  3. Perform computation
- Possibility to send different messages to different neighbors
- Identifiers of size $O(\log n)$
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Bit complexity (3 possible measures)

1. Total number of bits exchanged [Van Leeuwen et al.'87]
2. Max number of bits per edge [Schneider and Waterhofer'11]
3. Bit round complexity [Kothapalli et al.'06]
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$\rightarrow$ Measure 3 captures both time and communication complexities (silence is not free)

Best bounds

	Time	# Messages	Message size	# Bit rounds
Awerbuch'87	$O(n)$	$\boldsymbol{\Theta(\vert E\vert + n\log n)}$	$O(\log n)$ bits	$O(n\log n)$
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(in our setting: deterministic algorithms without knowledge)

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$\mathcal{STT}$	$O(D + \log n)$	$O((D + \log n)m)$	$O(1)$ bits	$\mathbf{\Theta(D + \log n)}$

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Lower bound — $\Omega(D + \log n)$ bit rounds

- $2\lceil \log_2((Id_{max} + 2)/3.5)\rceil = \Omega(\log n)$ bits — [Dinitz and Solomon'07]
- $\Omega(D)$ messages, even of size $O(\log n)$ — [Kutten et al.'15]

Algorithm $\mathcal{STT}$

The algorithm (no knowledge required on the graph)

1. A spreading algorithm that broadcasts the largest id to each node
2. A spanning tree algorithm that is associated to the spreading actions;
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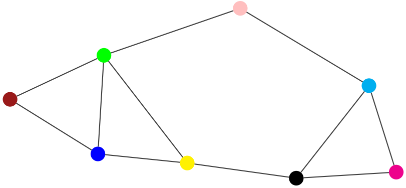

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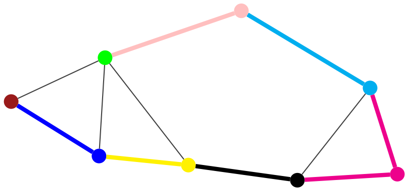

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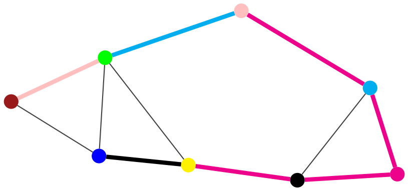

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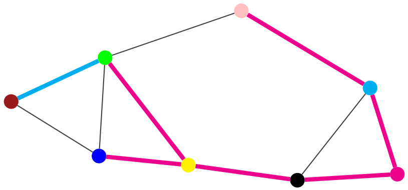

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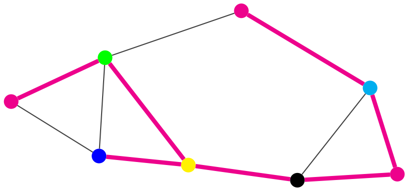

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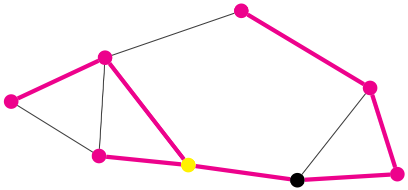

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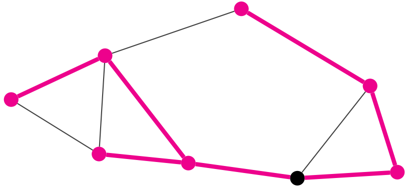

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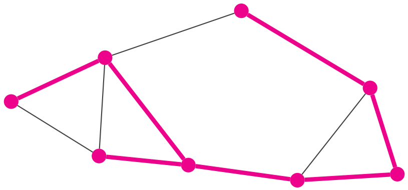

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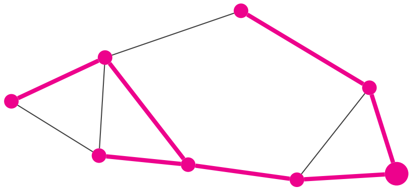

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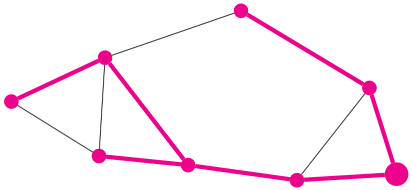

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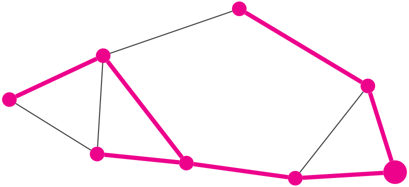

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$\rightarrow \mathcal{STT}$ takes it down to $O(D + \log n)$ bit rounds.

The main trick...

α-encoding of the identifiers

Given *Id*, we define $\alpha(Id)$ as the unary representation of the binary length of *Id*, followed by 0, followed by the binary representation of *Id*, i.e.:

$$\alpha(Id) = base1(|base2(Id)|) \cdot 0 \cdot base2(Id).$$

Ex. $Id = 25 \stackrel{2}{=} 11001$, then $\alpha(Id) = 11111011001$

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$\rightarrow$ Decidable based on prefixes, no need to wait...

The Spreading Algorithm $\mathcal{S}$

Pipelining the identifiers

Each node maintains the largest prefix of (encoded) identifier known so far, based on that of neighbors. Here are the round actions:

1. Update local largest prefix (e.g. based on that of neighbors)
2. Send signals indicating how the prefix was updated (or not)
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Exchanged signals (constant-size)

$signal \in \{append0, append1, delete1, delete2, delete3, change, null\}$
meaning that the largest known prefix was updated by:

- appending 0 or 1
- deleting one, two or three letters
- changing the last letter from 0 to 1
- leaving it unchanged

Update rules for prefix

The largest known prefix, denoted p here, is updated in each round based on current status (active or follower) and neighbors largest known prefixes (denoted p_n here) learnt through signals.

Update rules (in order of priority, only one per round)

Variables z, x, and y denote words.

(1.1) if $p = zx$ and $\exists p_n = z$ with delete signal, then delete x from p (up to $\underline{3}$ letters)

(1.2) if $p = z0x$ ($x \neq \epsilon$) and $\exists p_n = z1y$, then delete $|x|$ letters from p

(2) If $p = z0$ and $\exists p_n = z1y$, then $p \leftarrow z1$ and *status* $\leftarrow$ *follower*

(3) If $p = z$ and $\exists p_n = z1x$, then append 1 to p

(4) If $p = z$ and $\exists p_n = z0x$, then append 0 to p

(5) If *active*, then append the next bit of $\alpha(id)$ to p

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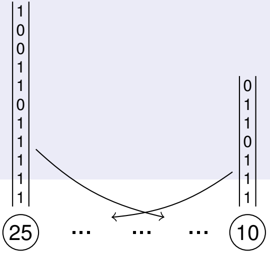

Resulting properties

Lemma 8

Let u and v be two neighbors and p_u and p_v their prefixes. The only possible cases after a round (up to renaming) are:

1. $p_u = p_v$
2. $p_u = p$ and $p_v = pw$ with $1 \leq |w| \leq 2$
3. $p_u = p0$ and $p_v = p1a$
4. $p_u = p1$ and $p_v = p0w$ and $|w| \leq 3$
5. $p_u = p$ and $p_v = pw$ and $3 \leq |w| \leq 6$ and u performed a delete

where p and w are words and a is either 0 or 1.

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After at most $6D + |\alpha(Id_{max})|$ rounds, the spreading is complete and all nodes know $\alpha(Id_{max})$.

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Spanning tree construction ($\rightarrow \mathcal{ST}$)

Occurs in parallel of spreading

- If u applies rule 2 relative to v, then v becomes u's parent.
- If u applies rule 3 relative to v, then v becomes (or remains) u's parent.
- If u applies rule 4 relative to v, then v becomes (or remains) u's parent.

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$\rightarrow$ Additional signals (constant-size).

Termination ($\rightarrow \mathcal{STT}$)

Recursive termination proceeds from the leaves up to the root, according to the following rule.

Termination rule

If it holds that

1. u is a follower
2. u's prefix is well-formed and identical to that of neighbors
3. for every child v of u, $term_v = \texttt{true}$
4. $term_u = \texttt{false}$

then $term_u \leftarrow \texttt{true}$ and u notifies its parent

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Remark: $term_u$ may wrongly be `true` due to local maxima and then re-become `false` (this is fine...). However, when an active node is notified by all its children, it becomes elected and correctly decides termination.

Conclusion

$\mathcal{STT}$ runs in $\Theta(D + \log n)$ bit rounds

1. Spreading phase in $O(D + \log n)$
2. Spanning tree in parallel of spreading
3. Termination detection costs $O(D)$ additional rounds

\+ Lower bound $\Omega(D + \log n)$

Open questions

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