Random walks (and Markov chains)

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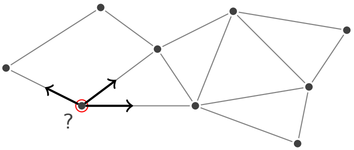
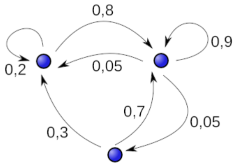
Random Walk





The basic version:

- Choose a neighbor at random
- Move to it
- Repeat

General version:

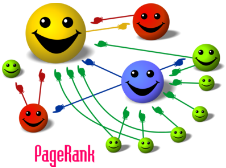
- Same, but...
- The graph may be directed
- Outgoing probabilities are unequal (sum to 1)

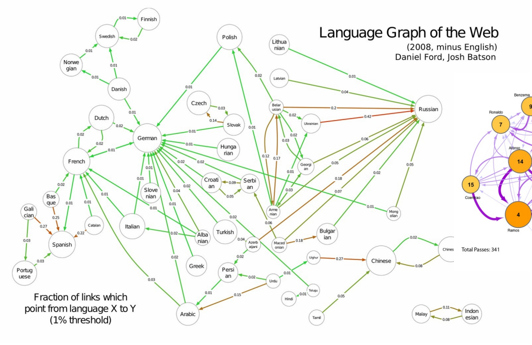
Main concepts

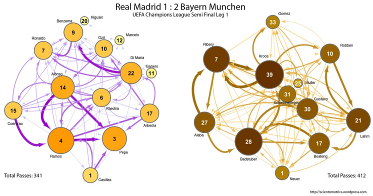
- **Hitting** time: How long until a given vertex is visited from another (on expectation or *w.h.p.*)
- **Cover** time: How long until all the vertices are visited (from the worst starting vertex)
- **Stationary** distribution: Probability to be on any vertex after infinite time
- **Mixing** time: How long before the distribution converges to stationary (up to some error)
- **Return** time: How long before we return to the starting vertex
- **Recurrent** / **transient** / **absorbing**: a vertex/state is recurrent if the probability to return to it is 1. It is transient otherwise. It is absorbing if we don't leave it (no outgoing edges).

Applications

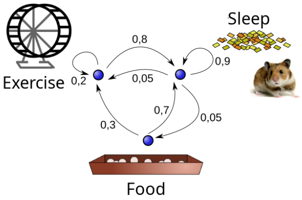
- **Finance**: Stock Market Analysis
- **Physics**: Brownian Motion, Diffusion
- **Computer science**: Graph traversal, Search Engines (→), Analysis of algorithms
- **Biology**: Animal movements, Spread of disease, Genetic evolution
- **Math**: Monte Carlo simulation, Laplace's equation







Markov process



Transition Matrix:

$$M = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 \end{bmatrix}$$

Probability distribution for the hamster: $\mathbf{x} = [\ p_s \quad p_f \quad p_e\ ]$ (with $p_s + p_f + p_e = 1$)

Let $\mathbf{x}_0 = [\ 1 \quad 0 \quad 0\ ]$ (initially sleeping), what about the next step?

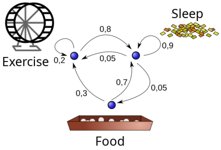
$$\mathbf{x}_1 = \mathbf{x}_0 M = [\ 1 \quad 0 \quad 0\ ] \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 \end{bmatrix} = [\ 0.9 \quad 0.05 \quad 0.05]$$

$$\mathbf{x}_2 = \mathbf{x}_1 M = \mathbf{x}_0 M^2 = [\ 1 \quad 0 \quad 0\ ] \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 \end{bmatrix}^2 = [\ 0.885 \quad 0.045 \quad 0.07]$$

$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ is a called **Markov chain**, and M is a **Markov model**.

Observation: $\mathbf{x}_i$ depends only on $\mathbf{x}_{i-1}$, not on earlier information (Markovian property)

Stationary distribution



How to compute the stationary distribution?

We know that $\mathbf{x}_{i+1} = \mathbf{x}M$.

$\rightarrow$ we are looking for $\mathbf{s}$ such that $\mathbf{s}M = \mathbf{s}$

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$$

$$\begin{array}{cccc} 0.9s_1 & +0.7s_2 & +0.8s_3 & = s_1 \\ 0.05s_1 & +0s_2 & +0s_3 & = s_2 \\ 0.05s_1 & 0.3s_2 & 0.2s_3 & = s_3 \\ s_1 & +s_2 & +s_3 & = 1 \end{array}$$

$$\begin{array}{cccc} 0.1s_1 & -0.7s_2 & -0.8s_3 & = 0 \\ 0.05s_1 & -s_2 & & = 0 \\ 0.05s_1 & 0.3s_2 & -0.8s_3 & = 0 \\ s_1 & +s_2 & +s_3 & = 1 \end{array}$$

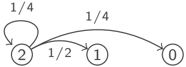
$\implies \mathbf{s} = [\ 0.884 \quad 0,0442 \quad 0,0718\ ]$

The hamster spends $\sim 88.4\%$ of its time sleeping.

Remark: $\mathbf{s}$ is the eigen vector of M for which the eigen value is 1. In the particular case that the graph is unweighted and undirected, each value s_i for vertex v_i is actually $d(v_i)/2m$ (thus, proportional to the degree).

Analysis of algorithms

Algorithm 1: We want to run an election among n candidates. In each step, each candidate flips a coin. The players having `tails` are eliminated. The players having `heads` play another round. The algorithm succeeds if at some point only one candidate remains.

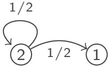


What is the probability that the algorithm succeeds if $n = 2$?

Let P_n be the probability of success if there are n candidates.

$$P_0 = 0$$
$$P_1 = 1$$
$$P_2 = \frac{1}{4}P_2 + \frac{1}{2}P_1 + \frac{1}{4}P_0$$
$$\implies \frac{3}{4}P_2 = \frac{1}{2} \implies \boxed{P_2 = \frac{2}{3}}$$

Algorithm 2: Same algorithm, but we restart it from scratch when it fails.



Now, the probability of success is 1. But how long does it take **on expectation**? (# coin flips)

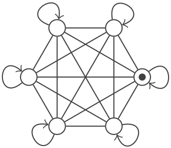
Let T_n be the remaining time if there are n candidates.

$$T_1 = 0$$
$$T_2 = 1 + \frac{1}{2}T_2 + \frac{1}{2}T_1 \implies \frac{1}{2}T_2 = 1 \implies \boxed{T_2 = 2}$$

Remark: We just analysed the **expected hitting time** from vertex 2 to vertex 1 in this graph!

Complete graph

Let's consider a random walk in a complete graph (with possibility to stay on the same vertex)

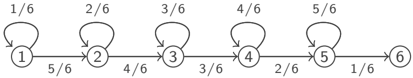


Expected **hitting** time from the stating vertex to a given vertex?

$\rightarrow$ In each step, $1/n$ chances of success
$\rightarrow$ expected time of n.

Expected **cover** time?

Markov model:
(States = #visited)



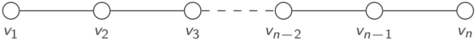
Remark: **cover** time in the original graph = **hitting** time from 1 to 6 in the second graph.

If k vertices remain to visit, what is the probability to move to a new vertex? Answer: k/n.

$\rightarrow$ Expected time to progress: n/k.

Total time: $\sum_{k\in[1,n-1]} \frac{n}{k} = n \cdot \sum_{k\in[1,n-1]} \frac{1}{k} = \Theta(n \log n)$ ("harmonic" numbers)

Path graph



Expected **return** time of v_n?

Recall that stationary distribution of a vertex v is $\frac{d(v)}{2m}$.
Theorem: **return** time is the inverse of that.
→ Return time of v_n is $2m = 2(n-1) = 2n-2$.

Expected **hitting** time from v_1 to v_n?

$H(v_1 \to v_n) = H(v_1 \to v_2) + H(v_2 \to v_3) + \cdots + H(v_{n-1} \to v_n)$ (linearity of expectation)

How much is $H(v_{n-1} \to v_n)$?
→ One less than the return time of v_n (why?)
→ $2n-3$

How much is $H(v_{n-2} \to v_{n-1})$?
→ Same argument, in a graph without v_n (why?)
→ $2n-5$

→ $H(v_1 \to v_n) = 1 + 3 + \cdots + (2n-5) + (2n-3)$
→ $H(v_1 \to v_n) = (n-1)^2 = \Theta(n^2)$.

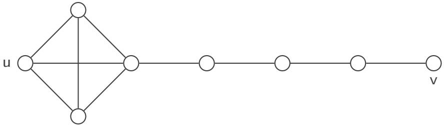
Expected **cover** time of the graph?

→ at most twice the worst hitting time (why?)
→ the cover time of the path graph is $\Theta(n^2)$

Worst cases

Undirected graphs:

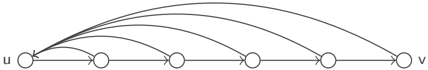
→ Lollipop graph (a $n/2$-clique, connected to a $n/2$-path).



→ Expected **hitting** time from u to v is $\Theta(n^3)$.

Directed graphs:

→ A directed path + return to origin in each vertex



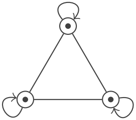
→ Expected **hitting** time from u to v is $\Theta(2^n)$

Coalescing time

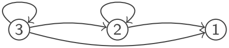
A token on every vertex. Each token performs a random walk. When two tokens meet, they merge.

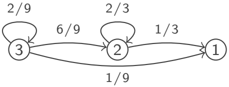
Coalescing time: How long until a single token remains?

Example: on a complete graph with loops:



Markov model:





Coalescing time of the original problem = **hitting** time from 3 to 1 in this graph.

3 choices for three tokens $\rightarrow$ 27 combinations of choices:

- 6 don't merge. $6/27 = 2/9$
- 18 merge two tokens. $18/27 = 6/9$
- 3 merge three tokens. $3/27 = 1/9$

3 choices for two tokens $\rightarrow$ 9 combinations of choices:

- 6 don't merge. $6/9 = 2/3$
- 3 merge. $3/9 = 1/3$

$T_1 = 0$

$T_2 = 1 + \frac{2}{3} T_2 + \frac{1}{3} T_1$

$T_3 = 1 + \frac{2}{9} T_3 + \frac{6}{9} T_2 + \frac{1}{9} T_1$

$\frac{1}{3} T_2 = 0 \implies T_2 = 3$

$\frac{7}{9} T_3 = 1 + \frac{6}{9} T_2 \implies \frac{7}{9} T_3 = 3 \implies \boxed{T_3 = 27/7}$