Temporal graphs (Spring 2024)

2. Basic definitions on temporal graphs

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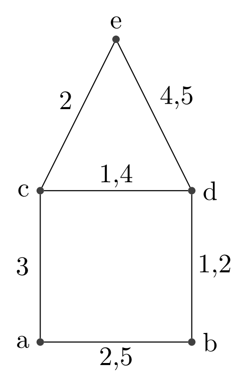
A temporal graph is a graph whose set of vertices and edges vary over the time. They model various phenomena in the real world, such as social interaction, communication in mobile networks (robots, drones, etc.), scheduling problems, and evolving networks.

In this introductory course, we focus on a basic model, where only the edges vary. Furthermore, the edges are undirected, and time is discrete. This setting is sufficient for discussing most aspects of temporal graphs.

Here is an example:

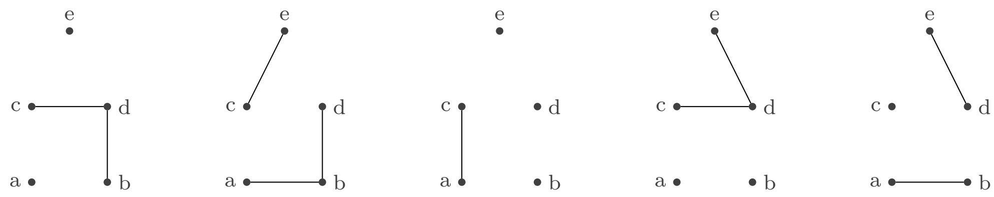


The edge labels indicate when these edges exist. For example, the edge ac is present only at time 3, the edge ab is present at times 2 and 5, and so on.

2.1 Definitions

Mathematically, a temporal graph can be defined as an **edge-labeled graph**: $\mathcal{G} = (V, E, \lambda)$, where V is a set of vertices, E is a set of edges ($E \subseteq V \times V$) and $\lambda : E \to 2^{\mathbb{N}}$ is a function that assigns to every edge of E a set of time labels. The graph (V, E) without time labels is called the **footprint** of $\mathcal{G}$.

It is sometimes useful to consider other points of views. A common one is as a **sequence of snapshots** $\mathcal{G} = \{G_1, G_2, ..., G_k\}$, where each snapshot G_i is the graph (V, E_i) with $E_i = \{e \in E \mid i \in \lambda(e)\}$. In other words, G_i contains the edges present at time i. The above graph then becomes:



A third point of view is to see the temporal graph as a **sequence of contacts**, where each contact is a pair (uv, t) such that $t \in \lambda(uv)$, i.e. the edge uv is present at time t. The same graph again corresponds to the following sequence:

$$\mathcal{G} = \{(cd, 1), (bd, 1), (ab, 2), (bd, 2), (cd, 2), (ac, 3), (cd, 4), (de, 4), (ab, 5), (de, 5)\}$$

Up to a few subtleties, all these representations are equivalent, and we will switch seemlessly among them depending on the context.

2.2 Temporal paths

Reachability in a temporal graph takes place over the time. Even if two nodes are never connected by a path (e.g. nodes a and e are never in the same component), they may still be able to reach each other using **temporal paths** – paths whose edges are taken in a chronological order. More formally, a temporal path in a graph $\mathcal{G} = (V, E, \lambda)$ is a sequence of contacts $\langle (e_i, t_i) \rangle$ such that $\langle e_i \rangle$ is a path in the footprint of $\mathcal{G}$, $\langle t_i \rangle$ is non-decreasing[1], and $t_i \in \lambda(e_i)$ for all i in the sequence. If $\langle t_i \rangle$ is increasing, then the temporal path is **strict**. Here are a few examples:

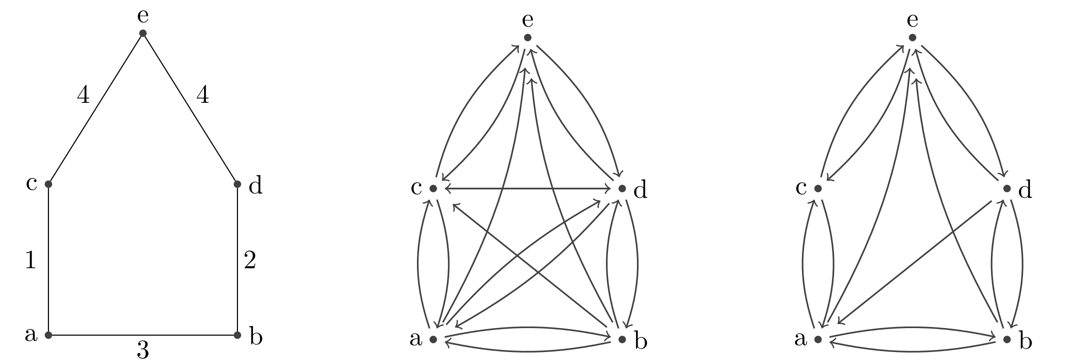
- $\langle (ec, 2), (ca, 3), (ab, 5) \rangle$ is a strict temporal path from e to b
- $\langle (ac, 3), (cd, 4), (de, 4) \rangle$ is a non-strict temporal path from a to e
- $\langle (ac, 3), (ce, 2) \rangle$ is not a temporal path

A common notation is $u \rightsquigarrow v$ to indicate that node u can reach node v.

2.3 Reachability graph

Given a temporal graph $\mathcal{G}$, one can construct a (static) directed graph $\mathcal{R}$ such that an arc exists from u to v in $\mathcal{R}$ if and only if a temporal path exists from u to v in $\mathcal{G}$. In other words, the graph $\mathcal{R}$ encodes the reachability relation among vertices of $\mathcal{G}$. One may consider two versions of this graph, depending on whether the temporal paths are required to be strict, as illustrated on fig. 1.

[1] In French, non-decreasing means "croissant", and increasing means "strictement croissant".



(a) Temporal graph $\mathcal{G}$ (b) Reachability of $\mathcal{G}$ (c) Strict reachability of $\mathcal{G}$

Figure 1: A temporal graph and its reachability graphs.

Unless otherwise mentioned, temporal paths are allowed to be non-strict. Intuitively, this comes to considering that information flows much faster than the network dynamics (indeed, at infinite speed). The following definitions can be adapted for either case.

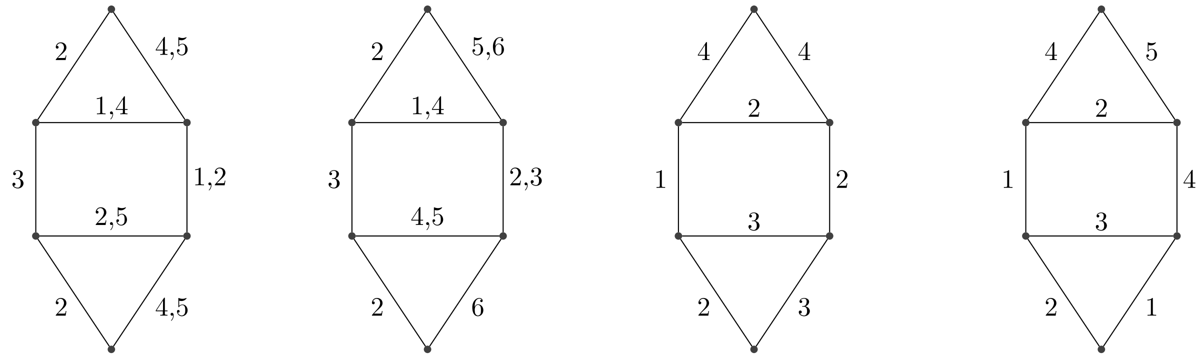
Connectivity. If all the nodes can reach each other, i.e. the reachability graph $\mathcal{R}$ is complete, then $\mathcal{G}$ is **temporally connected**. If a node can reach all the others, then this node is a **temporal source** (e.g. node a). If a node can be reached by all others, then it is a **temporal sink** (e.g. node e). Clearly, a graph is temporally connected if and only if all its vertices are temporal sources (or equivalently, all of them are temporal sinks).

2.4 Restricted settings

The definition of a temporal graph can be further restricted. The following two restrictions are commonly used.

- A temporal graph $\mathcal{G} = (V, E, \lambda)$ is **proper** if λ is locally injective, which means that two edges incident to a same node cannot have a common label.
- A temporal graph $\mathcal{G} = (V, E, \lambda)$ is **simple** if λ is single-valued, i.e. an edge can have only one presence time.

Properness is realistic in many contexts. For example, when the graph represents interactions taking place through pairwise phone calls. It also has the great advantage that the distinction between strict and non-strict temporal paths disappears. Simpleness is harder to motivate in terms of applications, but is convenient for studying the basic features of temporal graphs. General temporal graphs are typically hard to analyse mathematically, so understanding simpler models first is a good idea. An example of each kind is shown in fig. 2.



(a) Non-proper and non-simple.
(b) Proper and non-simple.
(c) Simple and non-proper.
(d) Simple and proper (= happy)

Figure 2: Properness and simpleness.

2.5 Exercises

2.5.1 Reachability (manually)

For each graph of fig. 2, indicate whether they are temporally connected (TC), and if not, whether they have at least one temporal source ($1 \rightsquigarrow *$) and one temporal sink ($* \rightsquigarrow 1$). We will consider the case that only <u>strict</u> temporal paths are allowed.

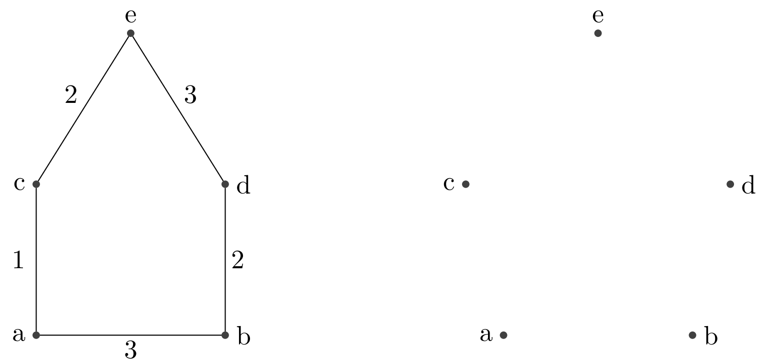
Graph	TC	$1 \rightsquigarrow *$	$* \rightsquigarrow 1$
a			
b			
c			
d			

2.5.2 Reachability (algorithmically)

When a temporal graph is proper, there exists an elegant algorithm for computing its reachability graph. Let's consider the following graph $\mathcal{G}$ that is both proper and simple:

1. Represent $\mathcal{G}$ as a sequence of contacts, ordered by time (breaking ties arbitrarily):

   $\mathcal{G} =$



(a) Temporal graph $\mathcal{G}$ (b) Reachability of $\mathcal{G}$

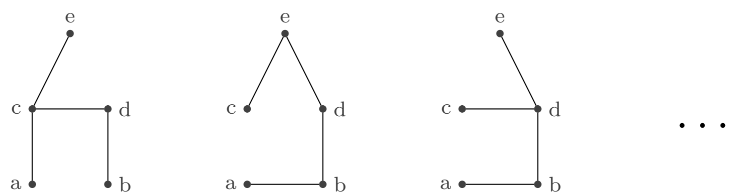
Figure 3: A temporal graph and its reachability graphs.

2. Try to draw (in fig. 3 (b)) the reachability graph of $\mathcal{G}$ by using this sequence in a single pass from left to right.

3. Write the pseudo code of the algorithm you discovered, in the format that you wish. (Better use a paper pen.)

2.5.3 Temporal graphs whose snapshots are connected graphs

An important family of temporal graphs, called AC (always connected) is when every snapshot is a connected graph. For example:



G_1 G_2 G_3

Assuming a temporal graph $\mathcal{G}$ on n vertices is in AC, what is the smallest time step k after which one can stop and declare that the prefix $\mathcal{G}_{[1,k]} = \langle G_1, G_2, \cdots, G_k \rangle$ is already temporally connected?

1. If non-strict temporal paths are allowed?

2. If only strict temporal paths are allowed? (justify your answer)